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Harmonic superconformal maps of surfaces in H^n

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Abstract

The class of harmonic superconformal maps from Riemann surfaces into real hyperbolic spaces is considered and harmonic sequences are constructed for these maps. They are used to obtain a rigidity result for such maps and to construct primitive lifts into an auxiliary flag space $\mathfrak{F}_m$. It is also shown that superconformal harmonic maps into H^{2m} and H^{2m-1} are locally described by 2D-affine Toda fields associated to the pair $(\mathfrak{so}(2m+1, C), \sigma)$, where σ is the involution determined by the non-compact real form $\mathfrak{so}(1, 2m)$. Applying the Adler–Kostant–Symes integration scheme to appropriate loop algebras we construct finite type primitive maps $\psi : H^2 \to \mathfrak{F}_m$, and harmonic superconformal maps $f : H^2 \to H^{2m}$ and hence finite type Toda fields. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, there has been a great deal of interest in the study of harmonic maps from surfaces to Riemannian symmetric spaces of compact type. The fact that the harmonic map equation for surfaces in these spaces is a kind of completely integrable system, was the starting point for the construction of harmonic maps from compact surfaces of genus one (i.e. two-tori) in compact symmetric spaces and Lie groups using ideas coming from soliton theory. There exist now a well established theory exposed in many articles, e.g. [4,5,7], to name only the most relevant to us.

When the target is a symmetric space of non-compact type, the integrable character of the harmonic map equation of surfaces still prevails as was proved by Bobenko [1] for surfaces

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in $\boldsymbol{R}^n$ and H^n. However, the maximum principle for (sub)harmonic functions implies that there are no non-constant harmonic maps from compact surfaces into these spaces.

From the point of view of Mathematical Physics the interest in harmonic maps of surfaces stems from its relation with σ-models. A classical solution of a σ-model is nothing but a harmonic map into some Riemannian manifold. The study of σ-models on non-compact (pseudo)-symmetric spaces arise for example in solid state physics and has many applications, see for instance [12] and the bibliography therein.

The goal of the present article is to begin the study of harmonic superconformal maps from non-compact connected Riemann surfaces into the the real n-dimensional hyperbolic space H^n of curvature -1. This class of maps can be regarded the natural counterpart of the harmonic superconformal maps of surfaces into Euclidean spheres S^n considered in [6]. Roughly speaking, a superconformal harmonic map is one whose harmonic sequence satisfies certain "orthogonality relations". In [6] a well-known soliton system called 2D-affine Toda field equations, were used by Bolton et al. to describe it superconformal harmonic maps of surfaces into CP^n and S^n. Let σ denote the conjugation authomorphism of $\mathfrak{so}(2m+1, \boldsymbol{C})$ respect to the non-compact real form $\mathfrak{so}(1, 2m)$. Along the paper we shall see that a 2D-affine Toda field associated with the pair $(\mathfrak{so}(2m+1, \boldsymbol{C}); \sigma)$, locally describes the geometry of harmonic superconformal maps of surfaces into H^{2m} and H^{2m-1}.

The paper is organized as follows. In Section 2, we recall some standard linear algebra and introduce some notation. We derive the harmonic map equation for conformal maps of surfaces into H^n. It is a semilinear elliptic equation which, as consequence of the maximum principle for subharmonic functions, it has only constant solutions if the domain is a compact surface. Hence, only non-compact surfaces are interesting. A brief Lie-algebraic introduction to generalized 2D-affine Toda field equations is also given.

In Section 3, we study complex vector subbundles E of the trivial bundle $M \times \boldsymbol{C}^{n+1} \to M$ over a Riemann surface M satisfying $E \cap E^{\perp_q} = \{0\}$ fiberwise, i.e. they are non-degenerate respect to the pseudo hermitian metric $q(z, w) = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k$ on $\boldsymbol{C}^{n+1}$. We show that these bundles have simple metric connections and hence can be equipped with a compatible holomorphic structure via the Koszul–Malgrange theorem [8]. Within this framework, we consider harmonic maps $f : M \to H^n$ for which their consecutive Hopf differentials $\eta_j = h^c((\partial^{(j)} f/\partial z^j), (\partial^{(j)} f/\partial z^j))\, dz^{2j},\ j = 1, \ldots, m-1$ vanish, except the last one η_m, where $m = [(n+1)/2]$. The vanishing of the first Hopf differential $\eta_1 = h^c(f_z, f_z)dz^2$ is equivalent to conformality of f, while the vanishig of η_2 is just conformality of the second fundamental form of f and so on. These conditions allow to generate inductively an ordered sequence of non-degenerate line subbundles of $M \times \boldsymbol{C}^{n+1}$:

$$L_{-m}, L_{-(m-1)}, \ldots, L_{-1}, L_0, L_1, L_2, \ldots, L_m, \quad \text{with} \quad \bar{L}_j = L_{-j},$$

satisfying orthogonallity relations $L_i \perp_q L_j$, $0 < |i-j| \le 2m$. Where the pseudohermitian metric q is positive on L_j for $|j| = 1, \ldots, m$ and negative on L_0. Besides there is a Gramm–Schmidt type algorithm which on every local chart (U, z) generates a meromorphic section f_j of L_j such that $\eta_j = h^c(f_j, f_j)dz^{2j}$. Our approach is conceptually analogous to that of [3,8], where (euclidean) harmonic sequences are constructed for harmonic maps of surfaces into compact Grassmanians and projective spaces.

Applying this construction we show in Proposition 3.4 that a harmonic superconformal map $f : M \to H^{2m}$ which lies fully into a totally geodesic H^{2m-1}, has a periodic sequence

$\{L_j\}$, i.e. $L_{j+2m} = L_j$, $j \in \mathbb{Z}$. This happens also when the target is the standard sphere S^{2m} (cf [3]). In Theorem 3.6, we show that a harmonic superconformal map $f : M \to H^{2m}$ is determined up to ambient isometries by the induced metric f^*h and the mth Hopf differential η_m.

In Section 4, the geometry of the harmonic sequence is used to construct primitive lifts of superconformal harmonic maps $f : M \to H^{2m}$ into an auxiliar flag domain $\mathfrak{F}_m$. This is a non-compact analog of some twistor constructions for harmonic maps into spheres and projective spaces (cf. [6]). Also special adapted frames or Toda frames for super conformal harmonic maps $f : M \to H^{2m}$ are considered and the 2D-affine Toda field equations associated with the pair $(\mathfrak{so}(2m+1, \boldsymbol{C}), \mathfrak{so}(1, 2m))$ are shown to be equivalent to the integrability conditions for the existence of a Toda frame.

Finally, in Section 5 after introducing some standard Loop-group machinery, we use the Adler–Kostant–Symes integration scheme to construct "finite type" solutions of the 2D-affine Toda field associated to $(\mathfrak{so}(2m+1, \boldsymbol{C}), \sigma)$. Here, in contrast with the compact target case (S^n, CP^n etc.) the solutions of the corresponding Toda equations are only local, i.e. they are not defined on the whole complex plane. Thus, modulo conformal maps of the plane, we obtain a recipe for the construction of finite type superconformal harmonic maps and primitive maps of the Poincare disc H^2. A related question is under what conditions a superconformal harmonic (respective primitive) map from H^2 is of finite type. This problem, however, seems to be more difficult and will be considered later.

2. Preliminaries

The n-dimensional Hyperbolic space H^n is the simply connected real space-form of constant sectional curvature -1. It can be realized as the unbounded sheet of the hyperboloid $h(x, x) = -1$ in $\boldsymbol{R}_1^{n+1}$ containing the point $e_0 = (1, 0, \ldots, 0)$, where $h(x, y) = -x_0 y_0 + \sum_{k=1}^n x_k y_k$, $x, y \in \boldsymbol{R}_1^{n+1}$.

The ambient Lorenzian inner product h induces on H^n a positive definite metric denoted with h and the Lie group $SO_o(1, n)$ acts transitively on H^n by isometries. In this way H^n becomes an homogeneous symmetric space of non-compact type. Taking the $SO_o(1, n)$-orbit of e_0 we have the representation $H^n = SO_o(1, n)/\{1\} \times SO(n)$.

For later use, we introduce the complex bilinear extension of h to $\boldsymbol{C}^{n+1}$ by $h^c(z, w) = -z_0 w_0 + \sum_{k=1}^n z_k w_k$. Defining $q(z, w) =: h^c(z, \bar{w})$, we obtain the Hermitian extension of h to $\boldsymbol{C}^{n+1}$ as well, hence $q(x, y) = h(x, y)\, \forall x, y \in \boldsymbol{R}_1^{n+1}$.

Recall that a matrix F is in $O(1, n)$ iff $FJF^T J^{-1} = I$, where $J = \mathrm{diag}(-1, 1, \ldots, 1)$. Thus, X is in $\mathfrak{so}(1, n) = Lie(O(1, n))$ iff $X^T J + JX = 0$. Hence, $\mathfrak{so}(1, n)$ is the set of matrices X of the form

$$X = \begin{pmatrix} 0 & b \\ b^T & A \end{pmatrix}, \quad b \in \boldsymbol{R}^n, A \in \mathfrak{so}(n)$$

On the other hand, the map

$$X \mapsto LX\bar{L} = \begin{pmatrix} 0 & ib \\ -ib^T & A \end{pmatrix}, \quad L = \mathrm{diag}(i, 1, \ldots, 1) \tag{2.1}$$

allows to think of $\mathfrak{so}(1, n)$ sitting inside $\mathfrak{so}(n+1, \boldsymbol{C})$. Hence, $\mathfrak{so}(1, n)^{\boldsymbol{C}} = \mathfrak{so}(n+1, \boldsymbol{C})$, and the conjugation authomorphism $\sigma : \mathfrak{so}(n+1, \boldsymbol{C}) \to \mathfrak{so}(n+1, \boldsymbol{C})$ respect to the non-compact real form $\mathfrak{so}(1, n)$ is given by

$$\mathfrak{so}(n+1, \boldsymbol{C}) \ni X = \begin{pmatrix} 0 & m \\ -m^{\mathrm{T}} & A \end{pmatrix} \mapsto \sigma(X) = J \cdot \bar{X} \cdot J = \begin{pmatrix} 0 & -\bar{m} \\ \bar{m}^{\mathrm{T}} & \bar{A} \end{pmatrix} \tag{2.2}$$

Notice that the map $F \mapsto LF\bar{L}$ embeds $\mathrm{O}(1, n)$ into $\mathrm{O}(n+1, \boldsymbol{C})$.

2.1. *The harmonic map equation*

Let M be a Riemannian manifold and $f : M \to H^n$ a smooth map. Then its second fundamental form is given by

$$\beta(U, V) = UVf - h(Uf, Vf)f - \mathrm{d}f(\nabla^M_U V), \quad U, V \in \mathfrak{X}(M)$$

The tension field of f is $\tau(f) = \operatorname{tr}\beta$.

If M is a Riemann surface (i.e. an oriented 2D manifold equipped with a conformal equivalence class of Riemannian metrics), and $f : M \to H^n$ a non-constant map, then f is called weakly-conformal if $h^{\mathrm{c}}(f_z, f_z) \equiv 0$ for every complex local coordinate z on M. A non-constant weakly-conformal harmonic map f is called a minimal branched immersion in the literature. Unless otherwise stated we will work with conformal maps $f : M \to H^n$ such that $\mathrm{d}f_x \neq 0 \, \forall x \in M$.

In the conformal case, the induced metric $g = f^*h$ can be written in a local complex coordinate (U, z) as $g = \rho^2 \mathrm{d}z \cdot \mathrm{d}\bar{z}$ for some smooth positive function $\rho : U \to \boldsymbol{R}$. Equivalently,

$$q(f_z, f_{\bar{z}}) = 0 \quad \text{and} \quad q(f_z, f_z) = \rho^2 \tag{2.3}$$

Proposition 2.1. *Harmonic map equation*: *let $f : M \to H^n$ be a conformal map from a Riemann surface M. Then $f : M \to H^n$ is harmonic and hence minimal if and only if on every complex chart $(z, U) \in M$ the following equation holds*

$$f_{z\bar{z}} = q(f_z, f_z)f \tag{2.4}$$

Proof. Let X be a tangent vector field on H^n, then the Levi-Civita connection ∇^H on H^n is given by

$$\nabla^H_{X_x} Y = X_x Y - h(X, Y)x \qquad \text{for every } Y \in \mathfrak{X}(H^n)$$

From this, we compute the second fundamental form of f,

$$\beta(U, V) = UVf - h(Uf, Vf)f - \mathrm{d}f(\nabla^M_U V) \qquad U, V \in \mathfrak{X}(M)$$

Respect to a complex chart we have $(||f_x||^4/4)\operatorname{tr}\beta = \beta(\partial/\partial z, \partial/\partial z)$, thus, (2.4) follows. □

The maximum principle for sub-harmonic functions, imply that the only harmonic maps from a compact Riemann surface M to H^n are the constant maps. Therefore, we will treat only non-compact Riemann surfaces.

2.2. 2D-affine Toda fields

They are systems of elliptic PDEs which exist in association with complex simple Lie algebras. To introduce these equations we recall some standard facts of Lie algebra theory.

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan sub algebra and $\Delta = \Delta(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{h})$ the corresponding root system. Choose a set of positive roots $\Delta^+ \subset \Delta$, and let $\pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta^+$ be the corresponding set of simple roots, where $r = \dim \mathfrak{h}$. Fix a maximal abelian subalgebra $\boldsymbol{t} \subset \mathfrak{h}$ so that $\boldsymbol{t}^{\boldsymbol{C}} = \mathfrak{h}$.

Being $\mathfrak{g}$ simple, one has a distinguished maximal root μ and we set $\pi^* = \pi \cup \{-\mu\}$. Thus, π^* labels the sets of nodes of the extended Dynkin diagram of $\mathfrak{g}$. The generalized 2D-affine Toda field system associated to $\mathfrak{g}$ is defined by the following system of $r = \dim \mathfrak{h}$ non-linear PDEs.

$$2\Omega_{z\bar{z}} = \sum_{\alpha \in \pi^*} d_\alpha \mathrm{e}^{2\alpha(\Omega)} H_\alpha, \quad \Omega(z, \bar{z}) \in i\boldsymbol{t} \tag{2.5}$$

where $H_\alpha \in i\boldsymbol{t}$ is the dual of α respect to the Killing form of $\mathfrak{g}$, and d_α are real arbitrary constants. Solutions of (2.5) are called Toda fields.

If σ denotes the involution of $\mathfrak{so}(2m+1, \boldsymbol{C})$ determined by the non-compact real form $\mathfrak{so}(1, 2m)$, we shall see that a 2D-affine Toda field associated to the pair $(\mathfrak{so}(2m+1, \boldsymbol{C}); \sigma)$, locally describe the geometry of harmonic superconformal maps from a Riemann surface M into H^{2m} and H^{2m-1}.

3. Harmonic sequences

Let $T \to CP^n$ the tautological line bundle, i.e. the complex line bundle whose fiber at the point $x \in CP^n$ is the complex line x itself. By abuse of notation we denote also by T the restriction to $CH^n = \{[v] \in CP^n : q(v, v) < 0\}$. The hermitian ambient metric q induces on T a negative definite hermitian metric q_T and $\mathrm{Hom}(T, T^\perp)$ is equipped with the corresponding tensor product hermitian holomorphic structure. For such a structure there is a compatible connection D on $\mathrm{Hom}(T, T^\perp)$ induced by the flat connection on $CH^n \times \boldsymbol{C}^{n+1} \to CH^n$ in the following way:

$$(D_X\sigma)s = \pi_{T^\perp}(X(\sigma \circ s)) - \sigma(\pi_T(Xs)), \quad X \in TM,$$

where $s \in C^\infty(T)$ is a local smooth section of T, σ is a smooth section of $\mathrm{Hom}(T, T^\perp)$, and $\pi_{T^\perp}$ is the projection onto the q-orthogonal complement $T^\perp$ (note that $T \cap T^\perp = \{0\}$, hence, $\pi_{T^\perp}$ and π_T are well defined). There is also a connection-preserving biholomorphic isomorphism:

$$T^{(1,0)}CH^n \xrightarrow{\gamma} \mathrm{Hom}(T, T^\perp), \quad \gamma(X)s = \pi_{T^\perp}(Xs), \quad s \in C^\infty(T), X \in TM \tag{3.1}$$

which satisfies,

$$q_T \otimes q_{T^\perp}(\gamma z, \gamma w) = -\frac{c}{2} g(z, w), \quad z, w \in T^{(1,0)} CH^n$$

where $c < 0$ is the holomorphic sectional curvature of the Kähler metric g on CH^n. There is an equivalent description of the geometry of $CP^n_1 = \{[v] \in CP^n : q(v, v) > 0\}$. For details, we refer the reader to [8,9].

A vector sub bundle $E \subset M \times \boldsymbol{C}^{n+1}$ is *non-degenerate* respect to q, if $E_x \cap E_x^{\perp_q} = \{0\} \, \forall x \in M$. When E is a line subbundle, i.e. complex 1D, then E is non-degenerate iff $q(v, v) \neq 0$ for every $0 \neq v \in E$. Hence, when the base M is connected, the induced metric q_E is always definite with the same sign on M. Consequently, we call a line subbundle $E \subset M \times \boldsymbol{C}^{n+1}$ *positive* or *negative* if q_E is positive definite or negative definite, respectively on M.

Recall now the bijective correspondence between smooth maps $\varphi : M \to CP^n$ and smooth complex line subbundles of $M \times \boldsymbol{C}^{n+1}$ given by

$$\varphi \leftrightarrow E_0 = \varphi^*(T) \tag{3.2}$$

Under (3.2), a smooth map $\varphi : M \to CH^n$ corresponds to a non-degenerate negative line subbundle $E \subset M \times \boldsymbol{C}^{n+1}$. Analogously a smooth map $\varphi : M \to CP^n_1$ corresponds under (3.2) to a non-degenerate positive line subbundle $E \subset M \times \boldsymbol{C}^{n+1}$.

Non-degenerate vector subbundles $E \subset M \times \boldsymbol{C}^{n+1}$ are useful since they have metric-compatible connections. In fact, for such a vector sub bundle, there are unambiguously defined orthogonal projectors $\pi_{E_x} : \boldsymbol{C}^{n+1} \to E_x, x \in M$, hence, a well defined connection $\nabla_E = \pi_E \circ d$, i.e.

$$(\nabla_E)_X s = \pi_E(Xs), \qquad s \in C^\infty(E), X \in TM$$

Note that the flat connection on $M \times \boldsymbol{C}^{n+1} \to M$ is given by $\nabla_X s = Xs$ for $s : M \to \boldsymbol{C}^{n+1}$ and $X \in TM$.

Given a complex vector bundle V over a Riemann surface M with a metric-compatible connection ∇, the well-known theorem of Koszul–Malgrange [8] guarantees the existence of a unique holomorphic structure on V compatible with ∇. A local section s is holomorphic respect to this structure if and only if $\nabla_X s = 0$ for every $X \in T^{(0,1)}M$.

In what follows, we shall consider every non-degenerate vector sub bundle $E \subset M \times \boldsymbol{C}^{n+1}$ equiped with the induced metric q_E and the unique holomorphic structure determined by $\nabla_E = \pi_E \circ d$, via the theorem of Koszul–Malgrange. Hence, a local section $s \in C^\infty(E)$ is holomorphic if and only if $(\partial/\partial \bar{z})s$ is q-orthogonal to E for any complex coordinate (U, z) on M. Respect to a complex coordinate we denote $(\nabla_E)_{(\partial/\partial z)}$ and $(\nabla_E)_{(\partial/\partial \bar{z})}$ simply by $(\nabla_E)'$, and $(\nabla_E)''$, respectively. Being E is non-degenerate, there is also a well-defined operator $A_E : TM \otimes E \to E^\perp$ given by

$$A_E(X \otimes s) =: \pi_{E^\perp}(Xs), \quad s \in C^\infty(E) \tag{3.3}$$

Notice that with our previous definitions $A_E = \nabla_{E^\perp}$. According to the splitting of the complexified tangent bundle $TM^{\boldsymbol{C}}$ into $(1, 0)$ and $(0, 1)$ types, the second fundamental form decomposes relative to a complex chart (z, U) as $A_E = A'_E + A''_E$, where

$$A'_E(s) = \pi_{E^\perp}\left(\frac{\partial}{\partial z} s\right), \qquad A''_E(s) = \pi_{E^\perp}\left(\frac{\partial}{\partial \bar{z}} s\right) \tag{3.4}$$

Remark 3.1. We shall regard A'_E and A''_E simply as bundle maps $E \to E^{\perp}$. This of course presupposes working on a specific complex chart but, it is routine calculation to check that all the constructions that follow, are independent of the choice of such a chart.

Lemma 3.1. *Let E be non-degenerate vector subbundle of $M \times \boldsymbol{C}^{n+1}$. Then $(A'_E)^* = -A''_{E^{\perp}}$.*

Proof. Let s_1 and s_2 be smooth local sections of E and $E^{\perp}$, respectively. Then, respect to any complex coordinate z, $0 = q(\partial_z s_1, s_2) + q(s_1, \partial_{\bar{z}} s_2)$. Hence, $q(A'_E(s_1), s_2) = -q(s_1, A''_{E^{\perp}}(s_2))$ and the Lemma follows. □

Now let $\varphi : M \to CP^n$ be the unique smooth map determined by a non-degenerate line sub bundle E. Then, if E is negative, φ factors through CH^n and if E is positive φ factors through CP^n_1, respectively.

Proposition 3.2. *In either case φ is a harmonic map if and only if A'_E is a holomorphic section of* $\mathrm{Hom}(E, E^{\perp})$. *Equivalently, φ is harmonic if and only if A''_E is antiholomorphic.*

Proof. Let us prove the statement when E is negative since the positive case is analogous. Let $\varphi : M \to CH^n$ be the corresponding map. By definition φ is harmonic if and only if $\mathrm{tr}(\nabla\, \mathrm{d}\varphi) = 0$, where ∇ is the connection on $\mathrm{Hom}(TM, TCH^n)$ determined by the respective connections ∇^M and $\nabla^{\varphi} = \varphi^*(\nabla^{CH^n})$. Extending $\mathrm{d}\varphi$ to a complex linear map from $TM^{\boldsymbol{C}}$ to TCH^n, it follows that φ is harmonic if and only if on any complex coordinate (z, U),

$$\nabla^{\varphi}_{\partial_{\bar{z}}} \mathrm{d}\varphi(\partial_z) = 0 \tag{3.5}$$

or, in an equivalent way,

$$\nabla^{\varphi}_{\partial_z} \mathrm{d}\varphi(\partial_{\bar{z}}) = 0 \tag{3.6}$$

Up to identifications (3.4) and (3.1) we get $\mathrm{d}\varphi(\partial_{\bar{z}}) = A''_{E_{\varphi}}$ and $\mathrm{d}\varphi(\partial_z) = A'_{E_{\varphi}}$. The pull-back connection ∇^{φ} corresponds under (3.1) to ∇_E. Then φ is harmonic if and only if A'_E is a holomorphic section of $\mathrm{Hom}(E, E^{\perp})$. This is equivalent to $\nabla'_E A'_E = 0$. In other words, φ is harmonic if and only if

$$A'_E \circ \nabla''_E = \nabla''_{E^{\perp}} \circ A'_E \tag{3.7}$$

□

For a smooth map $f : M \to H^n$ define $\varphi_0 : M \to CP^n$ by $\varphi_0(x) = [f(x)]$, i.e. $\varphi_0(x)$ is the complex line in $\boldsymbol{C}^{n+1}$ determined by $f(x)$. Since $h(f, f) = -1$, we have

$$q(\lambda f, \lambda f) = -|\lambda|^2 < 0, \qquad \forall \lambda \in \boldsymbol{C}^*$$

Hence, φ_0 factors through the (open) sub manifold $CH^n \subset CP^n$.

Proposition 3.3. *Let $f : M \to H^n$ be a conformal map. Then f is harmonic iff φ_0 is harmonic.*

Proof. Let

$$L_0 = \{(x, v) \in M \times \boldsymbol{C}^{n+1} | v \in \varphi_0(x)\} = \varphi_0^*(T) \subset M \times \boldsymbol{C}^{n+1}$$

Then $h(f, f) = -1$ implies $q(f_z, f) = q(f_{\bar{z}}, f) = 0$ for any complex coordinate, thus, $\nabla''_{L_0}(f) = 0$. This says that f is a (global) holomorphic section of L_0. On the other hand, f being conformal harmonic and non-constant, it satisfies $f_{z\bar{z}} = q(f_z, f_z)f$ on every complex chart (z, U). This clearly implies that $\nabla''_{L_0^\perp} \circ A'_{L_0}(f) = 0$. Thus, A'_{L_0} is holomorphic, hence, φ_0 is harmonic.

Conversely, if $\varphi_0 = [f]$ is harmonic, then (3.7) holds, and since $h(f, f) = -1$, one gets $\nabla''_{L_0^\perp} \circ A'_{L_0}(f) = 0$. That is, $\nabla''_{L_0^\perp}(f_z) = 0$. Hence, $f_{z\bar{z}} \in L_0$, and there is a (local) complex function λ such that $f_{z\bar{z}} = \lambda f$. From this, we conclude that $f_{z\bar{z}} = q(f_z, f_z)f$, so is f harmonic. □

Let $f : M \to H^n$ be a non-constant conformal harmonic map, then by the Proposition 3.3 f is a global holomorphic section of L_0 and A'_{L_0}, A''_{L_0} are non-vanishing holomorphic and antiholomorphic sections of $\mathrm{Hom}(L_0, L_0^\perp)$, respectively. The zeros of A'_{L_0} and A''_{L_0} are isolated and there are unique line sub bundles (extending through the zeros) $L_1, L_{-1} \subset L_0^\perp$ satisfying

$$\mathrm{Im}(A'_{L_0}) \subseteq L_1, \qquad \mathrm{Im}(A''_{L_0}) \subseteq L_{-1}$$

Hence, L_1 and L_{-1} are positive line sub bundles of $M \times \boldsymbol{C}^{n+1}$ such that $\bar{L}_1 = L_{-1}$. For, on any complex chart $z : U \to \boldsymbol{C}$, the complex derivatives f_z and $f_{\bar{z}}$ are local smooth sections of L_1 and L_{-1}, respectively and (outside possibly of a closed subset of isolated points of U),

$$q(f_z, f_z) = q(f_{\bar{z}}, f_{\bar{z}}) > 0.$$

In particular, L_1 and L_{-1} are non-degenerate and they can be equipped with holomorphic structures determined by ∇_{L_1} and $\nabla_{L_{-1}}$ via the theorem of Koszul–Malgrange. Moreover, $L_0 \perp L_1$, and $L_0 \perp L_{-1}$ because of

$$h(f, f) = -1, \quad h(f_z, f) = 0, \quad h(f_{\bar{z}}, f) = 0 \tag{3.8}$$

The orthogonality of L_1 and L_{-1} follows from the vanishing of the first complex Hopf quadratic differential $\eta_1 =: h^{\mathrm{c}}(f_z, f_z)\,\mathrm{d}z^2$.

Summing up these simple facts, we conclude that $\{L_{-1}, L_0, L_1\}$ is a set of mutually q-orthogonal non-degenerate line sub bundles satisfying

$$\bar{L}_0 = L_0, \qquad \bar{L}_1 = L_{-1} \tag{3.9}$$

Let φ_1 and $\varphi_{-1} : M \to CP^n$ be smooth maps which are in one-to-one correspondence with L_1 and L_{-1}, respectively. Then by our preceeding considerations φ_1 and φ_{-1} factor through the (open) sub manifold $CP_1^n \subset CP^n$.

Being both inclusions $i : L_1 \hookrightarrow L_0^\perp$ and $j : L_0 \hookrightarrow L_1^\perp$ antiholomorphic, so is the composition $(j \circ A''_{L_0} \circ i) : L_1 \to L_1^\perp$. But this is precisely A''_{L_1}, therefore, φ_1 is harmonic. Similarly we see that $A'_{L_{-1}}$ is holomorphic.

We also can prove this more directly by computing,

$$\partial_{\bar z} f_1 = q(f_1, f_1) f = -\frac{q(f_1, f_1)}{q(f, f)} f \tag{3.10}$$

On the other hand, $f_{\bar z}$ is a local antiholomorphic section of L_{-1}. However, if we set

$$f_{-1} =: \frac{f_{\bar z}}{q(f_{\bar z}, f_{\bar z})} \tag{3.11}$$

Then f_{-1} is a meromorphic section of L_{-1} (i.e. it is holomorphic except on the zero-set of the denominator) which satisfies

$$A'_{L_{-1}}(f_{-1}) = f, \quad \partial_{\bar z} f = -\frac{q(f, f)}{q(f_{-1}, f_{-1})} f_{-1} \tag{3.12}$$

If A'_{L_1} does not vanish identically on M then the same is true for $A''_{L_{-1}}$ as a consequence of (3.9). Let L_2 and L_{-2} be unique line sub bundles of $L_1^{\perp}$ defined by

$$\mathrm{Im}(A'_{L_1}) \subseteq L_2, \quad \mathrm{Im}(A''_{L_1}) \subseteq L_{-2}$$

Thus, L_2 and L_{-2} are positive, non-degenerate and satisfy $\bar{L}_2 = L_{-2}$ since, on a complex chart (U, z) we set, (except possibly on the subset of U where $f_1 = 0$)

$$f_1 =: f_z, \quad f_2 =: \partial_z f_1 - \frac{q(\partial_z f_1, f_1)}{q(f_1, f_1)} f_1. \tag{3.13}$$

Hence, $f_2 = A''_{L_1}(f_1)$ is a local meromorphic section of L_2 such that $q(f_2, f) = 0$, by (3.8) and conformality of f. Notice that $q(f_2, f) = 0$ is equivalent to Re f_2 and Im $f_2 \in T_f H^n$. Then if $f_2 \not\equiv 0$, $q(f_2, f_2) > 0$ on an open dense subset of U. Hence, L_2 is positive and non-degenerate and the same is true for L_{-2} considering $\bar{f}_1$ and $\bar{f}_2$. Therefore, L_2 and L_{-2} have holomorphic structures determined by ∇_{L_2} and $\nabla_{L_{-2}}$ via the theorem of Koszul–Malgrange and the corresponding maps $\varphi_2, \varphi_{-2} : M \to CP^n$ factor through CP^n_1. Arguing as before, we see that φ_2 and φ_{-2} are harmonic maps into CP^n_1.

A direct calculation shows that

$$A''_{L_2}(f_2) = \partial_{\bar z} f_2 = -\frac{q(f_2, f_2)}{q(f_1, f_1)} f_1 \tag{3.14}$$

Defining

$$f_{-2} =: \frac{-\bar{f}_2}{q(\bar{f}_2, \bar{f}_2)}, \tag{3.15}$$

it easily follows that f_{-2} is a local meromorphic section of L_{-2} such that

$$A'_{L_{-2}}(f_{-2}) = f_{-1}, \quad A''_{L_{-1}}(f_{-1}) = \partial_{\bar z} f_{-1} = -\frac{q(f_{-1}, f_{-1})}{q(f_{-2}, f_{-2})} f_{-2} \tag{3.16}$$

Remark 3.2. Notice that $f_2 = \alpha(\partial_z, \partial_z)$ is the $(2, 0)$ part of the complexified second fundamental form of f. In particular, f is totally geodesic if and only if $f_2 \equiv 0$.

So far, we constructed a set of non-degenerate line sub bundles $\{L_i\}_{i=-2}^{2}$ which, as consequence of conformality and formulas (3.8), satisfy orthogonality relations,

$$L_i \perp L_j, \quad \text{for } 0 < |i-j| \leq 3, \quad \text{and } -2 \leq i, j \leq 2 \tag{3.17}$$

Note however that from (3.17) one can not deduce that $L_{-2} \perp L_2$.

The *second Hopf differential* $\eta_2 = h^{\mathrm{c}}(f_2, f_2)\mathrm{d}z^4$ measures the failure of L_{-2} and L_2 to be q-orthogonal. By (3.14) it follows that η_2 is a (globally defined) holomorphic complex quartic differential.

If $\eta_2 \equiv 0$, we can inductively set forth the process provided that successive Hopf differentials η_k vanish on M. So, after completing $k > 2$ inductive steps we have $\eta_1 \equiv \eta_2 \equiv \cdots \equiv \eta_{k-1} \equiv 0$, and on a complex local chart (U, z) in M we have recursively generated $\boldsymbol{C}^{n+1}$-valued maps by the algorithm $f_{j+1} = (\partial/\partial z) f_j - (q((\partial/\partial z) f_j, f_j)/||f_j||^2) f_j$, $j = 1, \dots, k-2$, which are positive, i.e. $||f_j||^2 > 0$ since by construction and the vanishing of the η_j, $q(f_j, f) = 0$ holds, thus, $\mathrm{Re}\, f_j, \mathrm{Im}\, f_j \in T_f H^n$, and since $f_j \not\equiv 0$, then $q(f_j, f_j) > 0$.

These local generated $\boldsymbol{C}^{n+1}$-valued maps f_j give rise to a finite sequence of non-degenerate line sub bundles

$$L_{-k}, L_{-k+1}, \dots, L_{-1}, L_0, L_1, \dots, L_{k-1}, L_k,$$

all which are positive except L_0 which is negative, satisfying q-orthogonality relations,

$$L_i \perp L_j, \quad \text{for } 0 < |i-j| \leq 2k-1.$$

If $\eta_k \not\equiv 0$, then its ceros are isolated since it is a holomorphic $2k$-differential which measures the orthogonality of L_{-k} and L_k.

By Lemma 3.1, there are holomorphic bundle maps $L_j \overset{A'_{L_j}}{\to} L_{j+1}$, and antiholomorphic bundle maps $L_j \overset{A''_{L_j}}{\to} L_{j-1}$ with $(A'_{L_j})^* = -A''_{L_{j+1}}$ with $A'_{L_j}(f_j) = f_{j+1}$, and $A''_{L_j}(f_j) = f_{j-1}$. Due to the correspondence $L_j \leftrightarrow \varphi_j$ there is a sequence of maps

$$\varphi_{-k}, \dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots, \varphi_k$$

which are harmonic into CH^n for $j = 0$, and into CP_1^n if $j \neq 0$.

Note that the finite sequence of $\boldsymbol{C}^{n+1}$-valued maps $f_{-k}, f_{-k+1}, \dots, f_{-1}, f = f_0, f_1, f_2, \dots, f_k$ is locally generated from f on a complex chart (z, U) by a Gramm–Schmidt type algorithm namely,

$$\begin{cases} \dfrac{\partial f_j}{\partial z} = f_{j+1} + \left(\dfrac{\partial}{\partial z} \log||f_j||^2\right) f_j \\ \dfrac{\partial f_j}{\partial \bar{z}} = -\dfrac{||f_j||^2}{||f_{j-1}||^2} f_{j-1} \end{cases} \tag{3.18}$$

where $||f_j||^2 = q(f_j, f_j) > 0$ for $|j| \neq 0$ and $||f||^2 = -1$. In particular, this shows directly that each f_j is a local meromorphic ssection of the corresponding line bundle L_j.

In this paper, we distinguish a class of maps caracterized by the non-vanishing of the last Hopf differential. This motivates the following.

Definition 3.1. Let $f : M \to H^n$ be a harmonic (non-constant) map and let $m = [(n+1)/2]$. We say that f is, *superconformal* if $\eta_1 \equiv \eta_2 \equiv \cdots \equiv \eta_{m-1} \equiv 0$, but $\eta_m \not\equiv 0$.

Remark 3.3. Using induction and formulas (3.18), it can be shown that the above definition does not depend on a particular local complex coordinate. This is left to the reader.A second class of harmonic maps $f : M \to H^n$ is obtained if one requires that $\eta_1 \equiv \eta_2 \equiv \cdots \equiv \eta_m \equiv 0$. These are called *isotropic* in the literature, and were studied in [9]. It can be shown that isotropic harmonic surfaces in H^n can be obtained projecting suitable complex curves into some auxiliar flag space. This will be the matter of a subsequent paper (see [10]).

According to the preceeding definition a harmonic superconformal map $f : M \to \boldsymbol{H}^2$ is precisely a non-conformal harmonic map, whereas a conformal harmonic map $f : M \to H^4$ is superconformal or *isotropic*, i.e. $\eta_2 \equiv 0$.

As in the case of the spheres S^n in [2], there are two types of superconformal harmonic maps into H^{2m} namely, those which are linearly full into H^{2m}, and those which are linearly full into some H^{2m-1} totally geodesic immersed in H^{2m}.

The second case is simpler and deserves separate attention.

Proposition 3.4. *Let $f : M \to H^{2m}$ be a harmonic superconformal map which is linearly full into some totally geodesic copy of H^{2m-1} immersed in H^{2m}. Then the sequence of line bundles L_k of f is periodic of period $2m$, i.e. $L_{k+2m} = L_k$ for every integer k.*

Proof. By hypothesis, there exist a positive vector $\boldsymbol{n} \in \boldsymbol{R}_1^{2m+1}$ such that f is full into $V \cap H^{2m}$, which is a copy of H^{2m-1} totally geodesic immersed into H^{2m} where $V = \boldsymbol{n}^{\perp}$. In particular, $L_k \subset V^C$ for every k and, $L_i \perp L_j$ for $0 < |i-j| \le 2m-1$. Hence, $L_{-m} \perp L_j$ for $j = -m+1, \ldots, m-1$. This forces $L_{-m} = L_m$, since by hypothesis $(\oplus_{k=-m+1}^{m-1} L_k) \oplus L_m = V^C$, hence the $2m$-periodicity of the sequence L_k follows. □

This does not occur however with the harmonic superconformal maps f which are linearly full into H^{2m}.

Applying (3.18) and induction we obtain the following formulae which are analogous as those obtained in [2] for S^n.

Proposition 3.5. *Let $f : M \to H^{2m}$ be a superconformal harmonic map. Then*

$$f_{-j} = (-1)^{j+1} \frac{\bar{f}_j}{||f_j||^2}, \quad \text{for } 0 \le j \le m \tag{3.19}$$

Here, we consider the following question: which invariants determine a superconformal harmonic map up to ambient isometries?

Theorem 3.6. *Let $f, \hat{f} : M \to H^{2m}$ be harmonic superconformal maps having the same induced metrics and the same mth Hopf differentials. Then there is an isometry Φ of H^{2m} such that $\Phi \circ \hat{f} = f$.*

Proof. We assume that f and $\hat{f}$ are full and introduce (globally defined) real forms $\gamma_j = \tau_j |\mathrm{d}z|^2$ and $\hat{\gamma}_j = \hat{\tau}_j |\mathrm{d}z|^2$, where $\tau_j = ||f_{j+1}||^2 ||f_j||^{-2}$ and $\hat{\tau}_j = ||\hat{f}_{j+1}||^2 ||\hat{f}_j||^{-2}$. Since by hypothesis $f^*h = \hat{f}^*h$ we have $\gamma_0 = \hat{\gamma}_0$, and $\gamma_{-1} = \hat{\gamma}_{-1}$. As a consequence of the Gramm–Schmidt type algorithm (3.18) on a complex chart (U, z) we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log ||f_j||^2 = \tau_j - \tau_{j-1}, \quad |j| = 0, \ldots, m$$

From this, we deduce that γ_{-1} and γ_0 determine γ_j for $|j| = 1, \ldots, m$. In particular, $\tau_j = \hat{\tau}_j$ for $|j| = 0, 1, \ldots, m$, hence, $||f_j|| = ||\hat{f}_j||$ for $|j| = 0, 1, \ldots, m$.

On the other hand, by hypothesis we know that

$$q(f_i, f_j) = q(\hat{f}_i, \hat{f}_j) = 0, 0 < |i - j| \leq 2m$$

$$q(f_i, \bar{f}_i) = q(\hat{f}_i, \bar{\hat{f}}_i) = 0, i = 1, \ldots, m - 1$$

$$q(f_m, \bar{f}_m) = q(\hat{f}_m, \bar{\hat{f}}_m)$$

But $\{f_j, \bar{f}_j : j = 0, \ldots, m\}$ spans $\boldsymbol{C}^{2m+1}$. Hence, there is a matrix $A = A(z, \bar{z}) \in SU(1, 2m)$ such that

$$\hat{f}_j = Af_j, \quad \bar{\hat{f}}_j = A\bar{f}_j, \quad j = 0, \ldots, m$$

Thus, $A = \bar{A}$ and using (3.18), we see that $(\partial/\partial z)A = (\partial/\partial \bar{z})A = 0$, i.e. A is a constant matrix in $S\mathrm{O}(1, 2m)$. One can suppose (composing with isometries of H^{2m} if necessary) that there is a point $p_0 \in M$ such that $f(p_0) = g(p_0) = e_0$, the first vector of the canonical basis. In particular, $Ae_0 = e_0$ thus $A \in S\mathrm{O}_o(1, 2m)$. □

4. Primitive lifts

4.1. The flag domain

Let $\mathfrak{F}_m$ be the set of ordered sequences $(\mathfrak{L}_0, \mathfrak{L}_1, \ldots, \mathfrak{L}_m)$, $\mathfrak{L}_{m+1}$ of mutually q-orthogonal complex lines in $\boldsymbol{C}^{2m+1}$, where $\mathfrak{L}_0$ is the complexification of a negative line in $\boldsymbol{R}_1^{2m+1}$, and $\mathfrak{L}_j; j = 1, \ldots, m$ are positive complex lines. The pseudo-orthogonal Lie group $S\mathrm{O}_o(1, 2m)$ acts transitively on $\mathfrak{L}_m$ in the usual way

$$g.(\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_m) = (g\mathfrak{L}_0, g\mathfrak{L}_1, g\mathfrak{L}_2, \ldots, g\mathfrak{L}_m)$$

Let $\mathfrak{L}_0^0 = \boldsymbol{C}e_0$ and $\mathfrak{L}_k^0 = \boldsymbol{C}(e_{2k-1} - ie_{2k})$, where e_k is the standard $k + 1$-basis vector of $\boldsymbol{C}^{2m+1}$. Then the isotropy of the point $o = (\mathfrak{L}_0^0, \mathfrak{L}_1^0, \ldots, \mathfrak{L}_m^0) \in \mathfrak{F}_m$ is the mD compact torus

$$T = \mathrm{diag}(1, R(\varphi_1), R(\varphi_2), \ldots, R(\varphi_m)) | \varphi_k \in \boldsymbol{R}\}; \quad R(\varphi) = \begin{pmatrix} \cos \varphi_j & -\sin \varphi_j \\ \sin \varphi_j & \cos \varphi_j \end{pmatrix}$$

Thus, we may identify $\mathfrak{F}_m$ with the quotient $SO_o(1, 2m)/T$, and there is a natural projection $P : \mathfrak{F}_m \to H^{2m}$ defined as follows. Given $\mathfrak{L} = (\mathfrak{L}_0, \mathfrak{L}_1, \dots, \mathfrak{L}_m) \in \mathfrak{F}_m$, we may pick q-unit basis vectors $V_j \in \mathfrak{L}_j$, $j = 1, \dots, m$ and put $V_j = (1/\sqrt{2})(v_{2j-1} - iv_{2j})$, for $j = 1, \dots, m$ and v_0, the unique vector in $\boldsymbol{R}_1^{2m+1}$ determined by the following two conditions:

$$\begin{cases} h(v_0, v_0) = -1 \\ \{v_0, v_1, v_2, \dots, v_{2m}\} \text{ is a positively oriented } h\text{-orthonormal basis of } \boldsymbol{R}_1^{2m+1} \end{cases}$$

Let $P(\mathfrak{L}) = v_0 \in H^{2m}$. Thus, P sends an element $[F] \in \mathfrak{F}_m$ to the 0th column F_0 of $F \in SO_o(1, 2m)$. According to this definition, a point $\mathfrak{L} \in \mathfrak{F}_m$ is determined by $P(\mathfrak{L})$ and $\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_{m-1}$.

The Killing form of $\mathfrak{so}(1, 2m)$ induces a $SO_o(1, 2m)$-invariant pseudo-Riemannian metric on $\mathfrak{F}_m$, namely,

$$\langle X, Y\rangle = \tfrac{1}{2}\mathrm{tr}(X.Y), \quad X, Y \in T_o(\mathfrak{F}_m)$$

With this metric the projection $P : \mathfrak{F}_m \to H^{2m}$ becomes a (pseudo)-Riemannian submersion since the metric h on H^{2m} is given by $h(X, Y) = (1/2)\mathrm{tr}(X.Y)$, for $X, Y \in T_{e_0}(H^{2m})$.

There is also a $2m$-symmetric structure on $\mathfrak{F}_m$ defined as follows. Let $\boldsymbol{t}$ be the Lie algebra of T:

$$\boldsymbol{t} = \{\mathrm{diag}(0, \theta(u_1), \dots, \theta(u_m))|u_k \in \boldsymbol{R}\}, \qquad \theta(u) = \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}$$

Set

$$\sigma_j[\mathrm{diag}(0, \theta(u_1), \dots, \theta(u_m))] = iu_j.$$

Then as positive simple roots we take $\pi_+ = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$, where $\alpha_1 = \sigma_1, \alpha_2 = \sigma_2 - \sigma_1, \dots, \alpha_m = \sigma_m - \sigma_{m-1}$, and $\mu = \sigma_m + \sigma_{m-1}$ is the highest root.

Now pick elements $Z_k \in i\boldsymbol{t}^m$ so that $\alpha_j(Z_k) = \delta_{jk}$ and put $Z = \frac{1}{2m}\sum_{k=1}^{m} Z_k$. Then $\tau = \mathrm{Ad}(\exp 2\pi \mathrm{i} Z)$ is an automorphism of $\mathfrak{so}(2m+1, \boldsymbol{C})$ such that $\tau^{2m} = Id$. It is easily checked that τ preserves the real form $\mathfrak{so}(1, 2m)$, and if we denote also by τ the corresponding automorphism induced on $SO_o(1, 2m)$ (note that τ is determined up to conjugation by an element of $SO_o(1, 2m)$), then $\tau(T) \subset T$ and hence τ determines an isometry $\tau : \mathfrak{F}_m \to \mathfrak{F}_m$ of order $2m$.

We compute $Z_k = i\,\mathrm{diag}(0, \theta(0), \theta(0), \dots, \theta(0), \theta(-1)^k, \dots, \theta(-1))$. From this one can write down the $2m$-symmetry $\tau = \mathrm{Ad}(2\pi \mathrm{i} Z)$ explicitly,

$$\tau = \mathrm{Ad}\left[\mathrm{diag}\left(1, R\left(\frac{\pi}{m}\right), R\left(\frac{2\pi}{m}\right), \dots, R\left(\frac{(m-1)\pi}{m}\right), R(\pi)\right)\right] \tag{4.1}$$

where, $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Example 4.1. Note that for $m = 2$ and 3 the symmetry τ is given, respectively by

$$\mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \quad \mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Let $\nu = \mathrm{e}^{\mathrm{i}\pi/m}$. We have, thus, a direct sum decomposition

$$\mathfrak{so}(2m+1, \boldsymbol{C}) = \oplus_{j \in Z_{2m}} \mathfrak{m}_j, \quad \mathfrak{m}_j = \{X \in \mathfrak{so}(2m+1, \boldsymbol{C}) | \tau(X) = \nu^j X\}$$

Recall now from (2.2) the conjugation atomorphism $\sigma : \mathfrak{so}(2m+1, \boldsymbol{C}) \to \mathfrak{so}(2m+1, \boldsymbol{C})$ respect to the non-compact real form $\mathfrak{so}(1, 2m)$. We need the following.

Proposition 4.1. *The automorphisms τ and σ commute, i.e. $\sigma \circ \tau = \tau \circ \sigma$.*

Proof. Easily follows from the definitions of τ and σ and is left to the reader. □

As a consequence the ν^j-eigenspaces $\mathfrak{m}_j$ satisfy

$$[\mathfrak{m}_j, \mathfrak{m}_k] \subset \mathfrak{m}_{j+k}, \ j, k \in \mathbb{Z}_{2m} \quad \sigma(\mathfrak{m}_j) = \mathfrak{m}_{-j}. \tag{4.2}$$

Note that since $\mathfrak{m}_0 = \boldsymbol{t}^{\boldsymbol{C}}$, $\sigma(\mathfrak{m}_0) = \mathfrak{m}_0$.

Moreover, each $\mathfrak{m}_k$ is $\mathrm{Ad}(T)$-invariant and determines an $S\mathrm{O}_o(1, 2m)$-invariant subbundle $[\mathfrak{m}_k]$ of the complexified tangent bundle $T(\mathfrak{F}_m)^{\boldsymbol{C}}$ of $\mathfrak{F}_m$, namely,

$$[\mathfrak{m}_k](xT) = \mathrm{Ad}(x)(\bar{L} \cdot \mathfrak{m}_k \cdot L), \quad x \in S\mathrm{O}_o(1, 2m),$$

where $L = \mathrm{diag}(i, \overbrace{1, \ldots, 1}^{2m})$. Hence, we have the following identification

$$T(\mathfrak{F}_m)^{\boldsymbol{C}} = \oplus_{k \in Z_{2m}} [\mathfrak{m}_k]$$

On the other hand, we know from the general theory [6] that $\mathfrak{m}_1$ decomposes into a direct sum,

$$\mathfrak{m}_1 = \sum_{k=0}^{m} \mathfrak{g}_{\alpha_k}, \tag{4.3}$$

where $\mathfrak{g}_{\alpha_k} \subset \mathfrak{so}(2m+1, \boldsymbol{C})$ is the α_k-root space, and $\alpha_0 = -\mu$. From this we obtain the following.

Lemma 4.2. *The subspace* $\mathfrak{m}_1 \subset \mathfrak{so}(2m+1, \boldsymbol{C})$ *consists of matrices X having the following form,*

$$X = \begin{pmatrix} 0 & Q_1 & 0 & 0 & \cdots & 0 \\ -Q_1^{\mathrm{T}} & 0_2 & Q_2 & 0 & \cdots & 0 \\ 0 & -Q_2^{\mathrm{T}} & 0_2 & Q_3 & \cdots & 0 \\ 0 & 0 & -Q_3^{\mathrm{T}} & 0_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & \cdots & \cdots & 0 & 0_2 & Q_m \\ 0 & \cdots & \cdots & 0 & -Q_m^{\mathrm{T}} & 0_2 \end{pmatrix} \tag{4.4}$$

where

$$Q_1 = a_1(-1\,, i), \quad Q_j = a_j \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}, \quad j = 1, \ldots, (m-1);$$
$$Q_m = \begin{pmatrix} a & b \\ ia & ib \end{pmatrix}, \quad a_j, a, b \in \boldsymbol{C}$$

According to [6] we introduce the following.

Definition 4.1. An element $X \in \mathfrak{m}_1$ is called *cyclic* if $a, b, a_1, b_2, \ldots, a_{m-1}, b_{m-1} \in \boldsymbol{C} - \{0\}$ and $|a| + |b| > 0$.

We refer the reader to Lemma 2.1 of [6] for the characterization of cyclic elements in terms of the homogeneous generators of the ring of $\mathrm{Ad}(SO(2m+1, \boldsymbol{C}))$-invariant polinomials.

Definition 4.2. A map $\psi : M \to \mathfrak{F}_m$ is called *τ-primitive* if $\mathrm{d}\psi(T^{(1,0)}M) \subset [\mathfrak{m}_1]$, and contains a cyclic element, (i.e. $\mathrm{d}\psi(T^{(1,0)}M)$ is not contained in any sub bundle obtained from $\mathfrak{m}_1$ by omitting one of the direct summands in (4.3)).

Given a super conformal harmonic map $f : M \to H^{2m}$, we may construct a lift $\hat{f} : M \to \mathfrak{F}_m$ as follows. Put $\mathfrak{L}_0 = L_0, \mathfrak{L}_j = L_j$ for $j = 1, \ldots, (m-1)$, and let $\mathfrak{L}_m$ be determined by the condition $P(\mathfrak{L}_0, \mathfrak{L}_1, \ldots, \mathfrak{L}_{m-1}, \mathfrak{L}_m) = f$. Then L_m lives in $\mathfrak{L}_m \oplus \bar{\mathfrak{L}}_m = [\oplus_{-m+1}^{m-1} L_j]^{\perp_q}$ and define

$$\hat{f} = (\mathfrak{L}_0, \mathfrak{L}_1, \ldots, \mathfrak{L}_{m-1}, \mathfrak{L}_m) \tag{4.5}$$

Theorem 4.3. *Let $f : M \to H^{2m}$ be a harmonic map. If f is super conformal then the lift $\hat{f} : M \to \mathfrak{F}_m$ given by* (4.5) *is primitive. Conversely, if $\psi : M \to \mathfrak{F}_m$ is primitive, then $f = P \circ \psi : M \to H^{2m}$ is a super conformal harmonic map such that $\hat{f} = \psi$.*

Proof. Assume that the harmonic map f is superconformal. Let $\{L_j\}_{j=-m}^{m}$, be the sequence of non-degenerate line bundles of f, and f_j the corresponding local section of L_j, generated on a local complex chart (U, z) by algorithm (3.18). Define $\boldsymbol{R}^{2m+1}$-valued functions $F_1, \dots, F_{2m}$ on U by

$$\begin{cases} f_j = \frac{1}{\sqrt{2}}||f_j||(F_{2j-1} - \mathrm{i}F_{2j}), & j = 1, \dots, (m-1) \\ f_m = aF_{2m-1} - \mathrm{i}bF_{2m}, & \text{with} \quad |F_{2m-1}||^2 = ||F_{2m}||^2 = 1 \end{cases} \tag{4.6}$$

where a and b are real valued functions on U such that $\eta_m|_U = (a^2 - b^2)\mathrm{d}z^{2m}$, hence $a^2 - b^2 \not\equiv 0$ on U and $||f_m||^2 = a^2 + b^2$. We can assume that $||f_j|| > 0$, $j = 1, \dots, m$ on U since f_j vanish only at isolated points.

Define $\boldsymbol{F} = (f, F_1, F_2, \dots, F_{2m-1}, F_{2m})$. Then by (4.6) and (3.18), it follows that $\boldsymbol{F}$ is $\mathrm{O}(1, 2m)$-valued on U and we assume that $\boldsymbol{F}$ is $S\mathrm{O}_o(1, 2m)$-valued on U. We have

$$\boldsymbol{F} \cdot o = (\boldsymbol{C}f, \boldsymbol{C}(F_1 - \mathrm{i}F_2), \dots, \boldsymbol{C}(F_{2m-1} - \mathrm{i}F_{2m}))$$

Put

$$\mathfrak{L}_0 = \boldsymbol{C}f, \quad \mathfrak{L}_j = \boldsymbol{C}(F_{2j-1} - \mathrm{i}F_{2j}), \quad j = 1, \dots, m \tag{4.7}$$

Note that $\mathfrak{L}_m \neq L_m$, but $L_m \subset \mathfrak{L}_m \oplus \mathfrak{L}_m$. Define $\hat{f} = (\mathfrak{L}_0, \mathfrak{L}_1, \dots, \mathfrak{L}_m)$, hence, $\boldsymbol{F} \cdot o = \hat{f}$, i.e. $\boldsymbol{F}$ is a frame of $\hat{f}$ on U.

Now using (4.6) and (3.18) we see that $\boldsymbol{F}$ satisfies,

$$\boldsymbol{F}^{-1}\boldsymbol{F}_z = \bar{L}(\boldsymbol{A}_0 + \boldsymbol{A}_1)L,$$

where $\boldsymbol{A}_1 \in \mathfrak{m}_{-1}$, and $\boldsymbol{A}_0 \in \mathfrak{m}_0$.

In particular, $\hat{f}_z = \boldsymbol{F} \cdot \bar{L} \cdot \boldsymbol{A}_1 L \in [\mathfrak{m}_1](\boldsymbol{F} \cdot o) = \mathrm{Ad}(\boldsymbol{F})(\bar{L} \cdot \mathfrak{m}_1 \cdot L)$ where $\boldsymbol{A}_1 \in \mathfrak{m}_1$ is cyclic by (4.6) and by our assumption $||f_j|| > 0$ on U for $j = 1, \dots, m$. This shows that $\hat{f}$ is primitive.

Conversely, let $\psi : M \to \mathfrak{F}_m$ be a primitive map, $\psi = (\mathfrak{L}_0, \mathfrak{L}_1, \dots, \mathfrak{L}_m)$, and set $f = P \circ \psi$. Let (U, z) be a complex chart in M, and $\boldsymbol{F} = (F_0, F_1, \dots, F_{2m}) : U \to S\mathrm{O}_o(1, 2m)$ a local frame of ψ then,

$$\mathfrak{L}_0 = \boldsymbol{C}f, \quad \mathfrak{L}_j = \boldsymbol{C}(F_{2j-1} - \mathrm{i}F_{2m}), \quad j = 1, \dots, m$$

In particular, $\boldsymbol{F}$ is also a local frame for f, and being ψ primitive, $\boldsymbol{F}^{-1}\boldsymbol{F}_z = \bar{L}(\boldsymbol{A}_0 + \boldsymbol{A}_1)L$ with $\boldsymbol{A}_0 \in \mathfrak{m}_0$ and $\boldsymbol{A}_1 \in \mathfrak{m}_1$. Therefore, $\boldsymbol{A}_1$ has the form (4.4) where $a_1, \dots, a_{m-1}, a, b$, are complex functions on U vanishing only at isolated points. From the isotropy of $\mathfrak{L}_1$ follows that f is conformal and, from the structure of $\boldsymbol{A}_1$ we deduce that $\nabla''_{\mathfrak{L}_0^\perp} \circ A'_{\mathfrak{L}_0}(f) = 0$, hence, $f : M \to H^{2m}$ is harmonic. This also follows from the general theory in [7] since f is covered by a primitive map.

By rotating on the normal plane spanned by $\{F_1, F_2\}$ we can get $f_z = (||f_z||/\sqrt{2})(F_1 - \mathrm{i}F_2)$ or $a_1 = i||f_z||/\sqrt{2}$, (note that such a rotation leaves the complex line $\mathfrak{L}_1 = \boldsymbol{C}(F_1 - \mathrm{i}F_2)$ unchanged). Using the structure of $\boldsymbol{A}_1$ it is easy to check that $f_1 =: f_z$ is a holomorphic local section of $\mathfrak{L}_1$ and that $f_2 =: (\partial/\partial z)f_1 - (q((\partial/\partial z)f_1, f_1)/||f_z||^2)f_1$ defines an holomorphic local section of $\mathfrak{L}_2$. Performing a rotation on the normal plane spanned by $\{F_3, F_4\}$ if necessary, we can get (keeping the complex line $\mathfrak{L}_2$ unchanged), $f_2 = (||f_2||/\sqrt{2})(F_3 - \mathrm{i}F_4)$, and $a_2 = ||f_2||/2||f_1||$. Notice that $\eta_2 \equiv 0$ since $\mathfrak{L}_2$ is isotropic. Continuing this way and rotating appropriately (if needed) the vectors $\{F_{2j-1}, F_{2j}\}$, we generate inductively local holomorphic sections $f_j = (||f_j||/\sqrt{2})(F_{2j-1} - iF_{2j}) \in \mathfrak{L}_j$, and matrix coefficients $a_j = ||f_{j+1}||/2||f_j||$ for $j = 1, \dots, (m-1)$. Finally setting $f_m = (\partial/\partial z)f_{m-1} - (q((\partial/\partial z)f_{m-1}, f_{m-1})/||f_{m-1}||^2)f_{m-1}$, and using the structure of the matrix $\boldsymbol{A}_1$ once more, we see that the last coefficients in A_1 are given by

$$a = \frac{||f_{m-1}||^{-1}}{\sqrt{2}}\alpha, \quad b = \mathrm{i}\frac{||f_{m-1}||^{-1}}{\sqrt{2}}\beta,$$

where α, β are real functions on U such that $f_m = \alpha F_{2m-1} - \mathrm{i}\beta F_{2m}$ with $\alpha^2 - \beta^2$ vanishing only at isolated points. Define $L_m = \boldsymbol{C} \cdot f_m$. After a straightforward calculation, one concludes that f_m is a holomorphic section of L_m, and $\eta_m = (\alpha^2 - \beta^2)\mathrm{d}z^{2m}$ is a non-vanishing holomorphic $2m$-differential on U. Moreover, $L_m \subset \mathfrak{L}_m \oplus \overline{\mathfrak{L}_m}$. All this clearly says that f is a superconformal harmonic map into H^{2m} satisfying $\psi = \hat{f}$. □

4.2. *Special frames*

The proof of the Theorem 4.3 suggests that we may use the information contained in the harmonic sequence of a superconformal minimal map $f : M \to H^{2m}$ to construct special adapted frames of f.

In fact, let $f : M \to H^{2m}$ be superconformal harmonic, and $z : U \to \boldsymbol{C}$ a local complex coordinate on a simply connected open set $U \subset M$, then using (3.18) we generate positive $\boldsymbol{C}^{2m+1}$-valued $\{f_j\}|j| = 1, \dots, m$. Then by (3.18), $h^{\mathrm{c}}(f_m, f_m)$ is a non-vanishing holomorphic function on U and we may choose the local coordinate z so that $h^{\mathrm{c}}(f_m, f_m) = 1$ on U, hence z is unique up to arbitrary translations and multiplication by a $2m$ root of unity. A proof of this fact is given in [6]. We introduce real functions u_j on U by $||f_j|| = \mathrm{e}^{u_j}$ (recall that $||f_0||^2 = -1$) and $\boldsymbol{R}^{2m+1}$-valued functions $F_1, \dots, F_{2m-1}, F_{2m}$ on U by setting

$$f_j = \frac{\mathrm{e}^{u_j}}{\sqrt{2}}(F_{2j-1} - \mathrm{i}F_{2j}), \quad j = 1, 2, \dots, m-1$$

$$f_m = \cosh u_m(F_{2m-1}) - i \sinh u_m(F_{2m})$$

From (3.18) it follows that $\boldsymbol{F} = (f, F_1, \dots, F_{2m})$ is $SO(1, 2m)$-valued on U. According to (4.1) the change of the frame $\boldsymbol{F}$ on U is expressed by

$$\boldsymbol{F}_z = \boldsymbol{F}\bar{L} \cdot \boldsymbol{A} \cdot L \tag{4.8}$$

where $\boldsymbol{A}$ is given by

$$\boldsymbol{A}=\begin{pmatrix} 0 & Q_1 & 0 & 0 & \cdots & 0 \\ -Q_1^{\mathrm{T}} & \theta(-iu_1) & Q_2 & 0 & \cdots & 0 \\ 0 & -Q_2^{\mathrm{T}} & \theta(-iu_2) & Q_3 & \cdots & 0 \\ 0 & 0 & -Q_3^{\mathrm{T}} & \theta(-iu_3) & \cdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & \\ 0 & \cdots & \cdots & 0 & \theta(-iu_{m-1}) & Q_m \\ 0 & \cdots & \dots & 0 & -Q_m^{\mathrm{T}} & \theta(-iu_m) \end{pmatrix}, \tag{4.9}$$

$$Q_1=i\frac{\mathrm{e}^{u_1}}{\sqrt{2}}(1-i),\quad Q_j=-\frac{\mathrm{e}^{u_{j+1}-u_j}}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\ \text{for } j=1,\ \dots,\ (m-2),$$
$$Q_m=-\frac{\mathrm{e}^{-u_{m-1}}}{\sqrt{2}}\begin{pmatrix} \cosh u_m & -i\sinh u_m \\ i\cosh u_m & \sinh u_m \end{pmatrix}$$

Setting $\Omega=i\,\mathrm{diag}(0,\theta(-u_1),\theta(-u_2),\ \dots,\ \theta(-u_m))\in i\boldsymbol{t}$, then $\boldsymbol{A}$ can be decomposed as follows:

$$\boldsymbol{A}=\Omega_z+\mathrm{e}^{\Omega}\boldsymbol{M}\mathrm{e}^{-\Omega}\in\mathfrak{m}_0\oplus\mathfrak{m}_1, \tag{4.10}$$

where

$$\boldsymbol{M}=\begin{pmatrix} 0 & M_1 & 0 & 0 & \cdots & 0 \\ -M_1^{\mathrm{T}} & 0_2 & M_2 & 0 & \cdots & 0 \\ 0 & -M_2^{\mathrm{T}} & 0_2 & M_3 & \cdots & 0 \\ 0 & 0 & -M_3^{\mathrm{T}} & 0_2 & & \vdots \\ \vdots & \vdots & & & \ddots & \\ 0 & \dots & \dots & & 0_2 & M_m \\ 0 & \dots & \cdots & & -M_m^{\mathrm{T}} & 0_2 \end{pmatrix}\in\mathfrak{m}_1, \tag{4.11}$$

where

$$M_1=\frac{1}{\sqrt{2}}(i,1),\qquad M_j=\frac{1}{2}\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix},\quad j=1,\ \dots,\ (m-1),$$
$$M_m=\begin{pmatrix} -1 & 0 \\ -i & 0 \end{pmatrix} \tag{4.12}$$

Among (local) frames of a primitive map $\psi : M \to \mathfrak{F}_m$ we distinguish a special kind of frames. Following [6] we call a local frame $\boldsymbol{F} : U \to SO_o(1, 2m)$ a Toda frame if there is a complex coordinate $z : U \to \boldsymbol{C}$ and a smooth map $\Omega : U \to i\boldsymbol{t}$ such that

$$\boldsymbol{F}^{-1}\boldsymbol{F}_z = \Omega_z + \mathrm{Ad}(\mathrm{e}^{\Omega})\bar{L} \cdot \boldsymbol{M} \cdot L \in \mathfrak{m}_0 \oplus \bar{L} \cdot \mathfrak{m}_1 \cdot L \tag{4.13}$$

Note that $\bar{L} \cdot H \cdot L = H \,\forall H \in \mathfrak{m}_0$.

Remark 4.1. Using harmonic sequences we have produced a local Toda frame for a τ-primitive map $\psi : M \to \mathfrak{F}_m$ as can be seen from Eq. (4.10).

A Toda frame $\boldsymbol{F}$ is unique up to multiplication by an element of the compact torus $T \subset \mathrm{O}(1, 2m)$.

4.3. Integrability conditions

Here, we determine the integrability conditions for the existence of a (local) Toda frame of a primitive map $\psi : M \to \mathfrak{F}_m$. To this end we decompose the matrix $\boldsymbol{M} \in \mathfrak{m}_1$ into a sum of root matrices $X_j \in \mathfrak{so}(2m+1, \boldsymbol{C})$ defined by

$$X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 1 & \dots \\ -i & 0 & 0 & \cdots \\ -1 & 0 & 0 & \cdots \\ \vdots & 0 & & \end{pmatrix}, \qquad X_j = \frac{1}{2}\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0_2 & P_j & \dots \\ & & -P_j^{\mathrm{T}} & 0_2 & \dots \\ & & & & \ddots \end{pmatrix},$$

$$j = 2, \dots, m,$$

where

$$P_j = -E_{2j-3,2j-1} + iE_{2j-3,2j} - iE_{2j-2,2j-1} - E_{2j-2,2j},$$

0_2 represents the 2×2-zero matrix and $E_{\alpha,\beta}$ the square matrix with 1 in the (α, β) place and zeros outside. The last matrix X_0 is defined by

$$X_0 = \frac{1}{2}\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0_2 & P_0 \\ & & -P_0^{\mathrm{T}} & 0_2 \end{pmatrix},$$

$$P_0 = -E_{2m-3,2m-1} - iE_{2m-3,2m} - iE_{2m-2,2m-1} + E_{2m-2,2m}$$

Note that $\boldsymbol{C}X_j = \mathfrak{g}_{\alpha_j}$ is the root space corresponding to the simple root α_j. Also $\boldsymbol{M} = X_0 + X_1 + \cdots + X_m$ holds. Moreover,

$$\begin{cases} [X_j, \sigma(X_j)] = \mathrm{tr}(X_j \cdot \sigma(X_j)) \cdot H_j \\ [X_k, \sigma(X_l)] = 0, \ \text{for}\ k \neq l \end{cases} \tag{4.14}$$

where $H_j \in i\boldsymbol{t}$ are given by

$$\begin{aligned} H_1 &= \tfrac{1}{2} i \operatorname{diag}(0, \theta(-1), 0_2, \dots, 0_2) \\ H_j &= \tfrac{1}{2} i \operatorname{diag}(0, 0_2, \dots, \theta(1), \underbrace{\theta(-1)}_{j}, 0_2, \dots, 0_2),\ j = 2, \dots, m \\ H_0 &= \tfrac{1}{2} i \operatorname{diag}(0, 0_2, \dots, 0_2, \theta(1), \underbrace{\theta(1)}_{m}) \end{aligned} \tag{4.15}$$

We leave for the reader to check that $H_j \in i\boldsymbol{t}$ is dual to α_j respect to $\operatorname{tr} X \cdot Y$.

Our choice of matrices X_j is such that

$$d_j := \operatorname{tr}(X_j \cdot \sigma(X_j)) = \begin{cases} 2, & \text{for } j = 1 \\ -2, & \text{for } j = 0, 2, \dots, m \end{cases} \tag{4.16}$$

From (4.16) and (4.17) we obtain the following.

Lemma 4.4. Let $\bar{L} \cdot \boldsymbol{A} \cdot L = \boldsymbol{F}^{-1}\boldsymbol{F}_z$ where $\boldsymbol{A}$ is the matrix (4.10) and $\boldsymbol{B} = \sigma(\boldsymbol{A})$. Then the integrability condition $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{A}_{\bar{z}} - \boldsymbol{B}_z$ for the existence of a Toda frame $\boldsymbol{F}$ is equivalent to

$$2\Omega_{z\bar{z}} = \sum_{j=0}^{m} d_j \, e^{2\alpha_j(\Omega)} H_j \tag{4.17}$$

where $\Omega = i \operatorname{diag}(0, \theta(-u_1), \theta(-u_2), \dots, \theta(-u_m)) \in i\boldsymbol{t}$.

Proof. $\operatorname{Ad}(e^{\Omega})\boldsymbol{M} = \sum_{\alpha \in \pi^*} e^{\alpha_j(\Omega)} X_j$ by definition of X_j. Analogously we see that $\sigma(\operatorname{Ad}(e^{\Omega})\boldsymbol{M}) = \operatorname{Ad}(e^{-\Omega})\sigma(\boldsymbol{M}) = \sum_{\alpha \in \pi^*} e^{\alpha_j(\Omega)}\sigma(X_j)$. From this and (4.16) the Lemma follows. Note that $\sigma(\mathfrak{g}_{\alpha_j}) = \mathfrak{g}_{\alpha_{-j}}$. □

In terms of the real functions u_j Eq. (4.17) is equivalent to the following non-linear elliptic system

$$\begin{cases} 2u_{1z\bar{z}} = e^{2(u_2 - u_1)} + e^{2u_1} \\ 2u_{jz\bar{z}} = e^{2(u_{j+1} - u_j)} - e^{2(u_j - u_{j-1})}, \quad j = 2, \dots, m-2 \\ 2u_{(m-1)z\bar{z}} = e^{-2u_{m-1}} \cosh 2u_m - e^{2(u_{m-1} - u_{m-2})} \\ 2u_{mz\bar{z}} = -e^{-2u_{m-1}} \sinh 2u_m \end{cases} \tag{4.18}$$

Theorem 4.5. *Let (U, z) be a complex coordinate of M with $U \subset M$ a simply connected open subset and let $\Omega : U \to i\boldsymbol{t}$ be a smooth map. Then* Eq. (4.13) *has a real solution $\boldsymbol{F} \in \mathrm{SO}_o(1, 2m)$ if and only if the field $\Omega = i \operatorname{diag}(0, \theta(-u_1), \theta(-u_2), \dots, \theta(-u_m))$ satisfies the 2D-affine Toda field* Eqs. (4.17).

In this case $\psi = \pi \circ F : U \to \mathfrak{F}_m$ is *τ-primitive and $f(F)$ is a Toda frame of ψ.*

Proof. The integrability condition for the existence of a Toda frame $\boldsymbol{F}$ is $(\partial/\partial z)(\partial/\partial \bar{z})\boldsymbol{F} = (\partial/\partial \bar{z})(\partial/\partial z)\boldsymbol{F}$ which by (4.8) is equivalent to $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{A}_{\bar{z}} - \boldsymbol{B}_z$. This in turn is equivalent to system (4.18). The final statement follows from (4.10). □

Remark 4.2. Eq. (4.17) with real arbitrary constants d_j is called by some authors the generalized (elliptic) 2D-affine Toda field associated $\mathfrak{so}(2m+1, \boldsymbol{C})$ [11]. Our choice of constants d_j in (4.16) depend essentially on the matrices X_j and the conjugation authomorphism σ (2.2) determined by the non-compact real form $\mathfrak{so}(1, 2m)$. For this reason we call (4.17) the 2D-affine Toda field associated to the pair $(\mathfrak{so}(2m+1, \boldsymbol{C}); \sigma)$. Notice that the 2D-affine Toda fields considered in [6] are in fact associated to a pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is a complex simple Lie algebra and $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$.

Remark 4.3. Note that a Toda frame $\boldsymbol{F}$ with field Ω for which $u_m \equiv 0$ gives rise to a primitive map ψ which projects onto a superconformal harmonic map f which is full into a totally geodesic $H^{2m-1} \subset H^{2m}$.

5. Existence

Here, we apply the Adler–Kostant–Symes integration scheme to construct finite type primitive maps $\psi : H^2 \to \mathfrak{F}_m$ and harmonic superconformal maps $f : H^2 \to H^{2m}$. The main reference here is the expository article by Burstall and Pedit [7]. We introduce the standard ingredients of the general theory in order to keep the paper self-contained.

Given a primitive map $\psi : M \to \mathfrak{F}_m$, there is a lift $\boldsymbol{F} : \tilde{M} \to SO_o(1, 2m)$ defined on the universal covering space of M. Pulling back the left Maurer Cartan form θ of $SO(2m+1, \boldsymbol{C})$ by the frame $\boldsymbol{F}$, one obtains the $\mathfrak{so}(2m+1, \boldsymbol{C})$-valued one-form $\alpha = \boldsymbol{F}^{-1}\,\mathrm{d}\boldsymbol{F}$ on $\tilde{M}$ which splits into (1, 0) and (0, 1) types,

$$\alpha = \overbrace{A_1'\,\mathrm{d}z}^{\alpha_1'} + \overbrace{(A_0\,\mathrm{d}z + \bar{A}_0\,\mathrm{d}\bar{z})}^{\alpha_0} + \overbrace{A_1''\,\mathrm{d}\bar{z}}^{\alpha_1''}$$

where $A_0 = \bar{A}_0 : \tilde{M} \to \mathfrak{m}_0$, $A_1' : \tilde{M} \to \mathfrak{m}_1$, with A_1' cyclic and $A_1' = \sigma(A_1'')$. The compatibility (or integrability) condition for the existence of such frame is given by the so called zero curvature condition or Murer–Cartan equation for $\boldsymbol{F}$:

$$\mathrm{d}\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0$$

In terms of the one-form $\alpha_0, \alpha_1', \alpha_1''$ the equation above decomposes into

$$\mathrm{d}\alpha_0 + \tfrac{1}{2}[\alpha_0 \wedge \alpha_0] = -[\alpha_1' \wedge \alpha_1''];\qquad \mathrm{d}\alpha_1' + [\alpha_0 \wedge \alpha_1'] = 0 \tag{5.1}$$

The key point is that these equations may be reinterpreted by introducing a complex spectral parameter $\lambda \in S^1$ in α, namely,

$$\alpha_\lambda =: \lambda\alpha_1' + \alpha_0 + \lambda^{-1}\alpha_1'',\quad \lambda \in S^1$$

Then $\alpha = \alpha_1' + \alpha_0 + \alpha_1''$ satisfies (5.1) if and only if the loop of one-form α_λ satisfies

$$\mathrm{d}\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad \forall\lambda \in S^1 \tag{5.2}$$

Integrating (5.2) on the simply connected $\tilde{M}$, we obtain an S^1-loop of frames $\boldsymbol{F}_\lambda : \tilde{M} \to SO_o(1, 2m)$ such that

$$\boldsymbol{F}_\lambda^{-1}\,\mathrm{d}\boldsymbol{F}_\lambda = \alpha_\lambda, \quad \boldsymbol{F}_\lambda(p_0) = Id \quad \forall\lambda \in S^1$$

where $p_0 \in \tilde{M}$ is some fixed point.

Hence, one gets an S^1-loop of τ-primitive maps $\psi_\lambda = \pi \circ \boldsymbol{F}_\lambda : \tilde{M} \to \mathfrak{F}_m$ with $\psi_1 = \psi$ where $\psi_\lambda(p_0) = \psi(p_0)\,\forall\lambda \in S^1$. In particular, this shows that primitive maps and hence harmonic superconformal maps exist in S^1-loops that are called associated families in [7].

In order to get a different perspective of the above situation, we introduce here some Loop algebras and groups following [7]. Let $\mathfrak{g}$ be the non-compact real form $\mathfrak{so}(1, 2m)$ of the complex simple Lie algebra $\mathfrak{so}(2m+1, \boldsymbol{C})$. Recall from (2.1) and (2.2) that $\mathfrak{g}^{\boldsymbol{C}} = \mathfrak{so}(2m+1, \boldsymbol{C})$. The twisted complex loop algebra is defined by

$$\Lambda\mathfrak{g}_\tau^{\boldsymbol{C}} = \{\xi : S^1 \to \mathfrak{g}^{\boldsymbol{C}} : \xi(\mathrm{e}^{\mathrm{i}\pi m}\lambda) = \tau\xi(\lambda)\,\forall\lambda \in S^1\}$$

There is also the real form of $\Lambda\mathfrak{g}_\tau^{\boldsymbol{C}}$ determined by $\mathfrak{g}$:

$$\Lambda\mathfrak{g}_\tau = \{\xi \in \Lambda\mathfrak{g}_\tau^{\boldsymbol{C}} : \xi(\lambda) \in \mathfrak{g}\,\forall\lambda \in S^1\}$$

Any loop $\xi \in \Lambda\mathfrak{g}_\tau^{\boldsymbol{C}}$ has a Laurent expansion $\xi(\lambda) = \sum_{j\in\boldsymbol{Z}} \xi_j\lambda^j$ with $\xi_j \in \mathfrak{m}_j$, i.e. such that $\tau\xi_j = \nu^j\xi_j$ ($\nu = \mathrm{e}^{\mathrm{i}\pi/m}$). Given $\xi \in \Lambda\mathfrak{g}_\tau^{\boldsymbol{C}}$ its conjugate respect to the real form $\Lambda\mathfrak{g}_\tau$ is defined by

$$\sigma(\xi)(\lambda) = \sum \sigma(\xi_n)\bar{\lambda}^n,$$

where σ is the conjugation authomorphism in $\mathfrak{g}^{\boldsymbol{C}}$ determined by $\mathfrak{g}$, see (2.2). In particular,

$$\sum_{k\in\mathbb{Z}} \xi_k\lambda^k \in \Lambda\mathfrak{g}_\tau \text{ iff } \sigma(\xi_k) = \xi_{-k} \quad k \in \mathbb{Z}$$

Note that a map $\xi : S^1 \to \mathfrak{so}(2m+1, \boldsymbol{C})$ is in $\Lambda\mathfrak{g}_\tau$ iff it satisfies the following conditions:

$$\begin{aligned} &\sigma(\xi(\lambda)) = \xi(\lambda)\,\forall\lambda \in S^1 \\ &\tau(\xi(\lambda)) = \xi(\nu\lambda)\,\forall\lambda \in S^1 \end{aligned} \tag{5.3}$$

The first one is called the reality condition determined by the non-compact real form $\mathfrak{so}(1, 2m)$.

Define a subalgebra $\Lambda^-\mathfrak{g}_\tau^{\boldsymbol{C}} = \{\xi \in \Lambda\mathfrak{g}_\tau^{\boldsymbol{C}} : \xi = \sum_{k\le 0} \xi_k\lambda^k, \xi_0 \in \boldsymbol{t}\}$, then one has an orthogonal decomposition into complementary (closed) subalgebras,

$$\Lambda\mathfrak{g}_\tau^{\boldsymbol{C}} = \Lambda\mathfrak{g}_\tau \oplus \Lambda^-\mathfrak{g}_\tau^{\boldsymbol{C}}$$

with respect to the weakly non-degenerate invariant real symmetric bilinear form

$$(\xi, \eta) = \tfrac{1}{2}\,\mathrm{Im}\int_{S^1} \mathrm{tr}\,\xi\cdot\eta, \quad \xi, \eta \in \Lambda\mathfrak{g}_\tau^{\boldsymbol{C}}$$

Let $\pi_K : \Lambda\mathfrak{g}_\tau^{\boldsymbol{C}} \to \Lambda\mathfrak{g}_\tau$ be the projection determined by (,) and the complement $\Lambda^-\mathfrak{g}_\tau^{\boldsymbol{C}}$. Then π_K is given by

$$\pi_K(\xi) = \xi^{(-)} + \xi_{\boldsymbol{t}}^0 + \sigma(\xi^{(-)}),$$

where $\xi^{(-)} = \sum_{j<0} \xi_j \lambda^j \in \Lambda^-\mathfrak{g}_\tau^{\boldsymbol{C}}$.

For a fixed $d \in \mathbb{N}$ such that $d \equiv 1 \operatorname{mod}(2m)$ consider the function,

$$f(\xi) = \frac{1}{4}\int_{S^1} \lambda^{1-d} \operatorname{tr} \xi^2 = f_1(\xi) + \mathrm{i} f_2(\xi), \tag{5.4}$$

Then projecting the gradients of f_i on $\Lambda\mathfrak{g}_\tau$, one obtains two Hamiltonian vector fields X_1, X_2 on $\Lambda\mathfrak{g}_\tau$.

The truncated Laurent polinomial loops of longitude d form a finite dimensional subspace of $\Lambda\mathfrak{g}_\tau$, namely,

$$\Lambda_d = \{\xi \in \Lambda\mathfrak{g}_\tau : \xi_n = 0\, \forall |n| > d\}$$

Restricting f_1, f_2 to Λ_d, then for $\xi \in \Lambda_d$,

$$\begin{aligned} X_1 &= [\xi, \pi_K(\nabla_\xi f_2)] \\ X_2 &= [\xi, \pi_K(\nabla_\xi f_1)], \end{aligned} \tag{5.5}$$

where $\nabla_\xi f_1 = \lambda^{1-d}\xi$, $\nabla_\xi f_2 = \mathrm{i}\lambda^{1-d}\xi$ are the gradients of f_1, f_2 respect to (,), respectively. Since f_i Poisson-commute [7], their commutator vanish, i.e. $[X_1, X_2] = 0$. Note that if $d \equiv 1 \operatorname{mod}(2m)$ then the gradients ∇f_j defined above are $\Lambda\mathfrak{g}_\tau^{\boldsymbol{C}}$-valued.

Both commuting Hamiltonian vector fields X_i considered above are of Lax form, i.e: $X(\xi) = [\xi, E(\xi)]$. Then if $\xi(t)$ is any integral curve of such an X, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\xi, \xi) = 2\left(\frac{\mathrm{d}}{\mathrm{d}t}\xi, \xi\right) = 2([\xi, E(\xi)], \xi) = -2(E(\xi), [\xi, \xi]) = 0$$

Note that the bilinear form (5.1) is non-definite on $\Lambda\mathfrak{g}_\tau$ and, thus, on Λ_d. Hence, $\xi(t)$ evolves in a generic non-euclidean sphere contained in Λ_d and, thus, may not be defined for every $t \in \boldsymbol{R}$.

Since $[X_1, X_2] = 0$ their flows X_1^s and X_2^t commute,

$$X_1^s \circ X_2^t = X_2^t \circ X_1^s$$

for s and t where they are defined. Fixed an initial condition $\xi_o \in \Lambda_d$, define $\xi : U \subset \boldsymbol{R}^2 \to \Lambda_d$ by

$$\xi(s,t) = X_1^s \circ X_2^t(\xi_o),$$

for all (s, t) in a suficiently small open neighbourhood of $(0, 0)$. Then one verifies that $\xi(s, t)$ satisfies

$$\mathrm{d}\xi = X_1(\xi)\,\mathrm{d}s + X_2(\xi)\,\mathrm{d}t, \quad \xi(0,0) = \xi_o$$

Now we define commuting complex vectorfields V and $\sigma(V)$ on $\Lambda_d^{\boldsymbol{C}}$ by $X_1 = V + \sigma(V)$, $X_2 = \mathrm{i}(V - \sigma(V))$, hence,

$$\begin{aligned} V(\xi) &= \left[\xi, \frac{\xi_{d-1}}{2} + \lambda\xi_d\right] \\ \sigma(V)(\xi) &= \left[\xi, \frac{\xi_{-d+1}}{2} + \lambda^{-1}\xi_{-d}\right] \end{aligned} \tag{5.6}$$

Introducing the complex variable $z = s + it$ and applying the above argument to it follows that the ODE's system:

$$\begin{cases} \dfrac{\partial}{\partial z}\xi = V(\xi), \\ \dfrac{\partial}{\partial z}\xi = \sigma(V)(\xi), \\ \xi(0,0) = \xi^{\circ}, \end{cases} \tag{5.7}$$

has a unique solution $\xi : U_0 \to \Lambda_d$ for any given initial condition $\xi^{\circ} \in \Lambda_d$, where $U_0 \subset \boldsymbol{R}^2$ is an open neighborhood of $(0,0) \in \boldsymbol{R}^2$ which we can assume to be connected and simply connected.

With this solution ξ the Λ_1-valued one-form

$$\alpha_\lambda = \lambda\xi_d \,\mathrm{d}z + \tfrac{1}{2}(\xi_{d-1}\,\mathrm{d}z + \xi_{-d+1}\,\mathrm{d}\bar{z}) + \lambda^{-1}\xi_{-d}\,\mathrm{d}\bar{z},$$

satisfies the zero curvature condition or Maurer-Cartan equation:

$$\mathrm{d}\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \; \forall\lambda \in S^1 \tag{5.8}$$

Thus, integrating on the simply connected U_0 we get an S^1-loop of frames $\boldsymbol{F}_\lambda : U_0 \to SO_o(1, 2m)$ with $\boldsymbol{F}_\lambda^{-1}\,\mathrm{d}\boldsymbol{F}_\lambda = \alpha_\lambda \; \forall\lambda \in S^1$ and one can arrange the constants of integration so that $\boldsymbol{F}_\lambda(0,0) = Id \, \forall\lambda \in S^1$.

5.1. Integration of the Toda equations

Using the above construction it is possible to get solutions of the 2D-affine Toda fields (4.17) associated to the pair $(\mathfrak{so}(2m+1, \boldsymbol{C}); \sigma)$ as follows.

Inspecting the ODE system (5.7) we see that the λ^{d-1} and λ^d terms satisfy

$$\frac{\partial}{\partial z}\xi_{d-1} = [\xi_d, \xi_{-d}], \qquad \frac{\partial}{\partial z}\xi_d = \frac{1}{2}[\xi_d, \xi_{1-d}], \qquad \frac{\partial}{\partial z}\xi_d = -\frac{1}{2}[\xi_d, \xi_{d-1}]. \tag{5.9}$$

In particular, the $\mathfrak{m}_0$ part of $\boldsymbol{F}^{-1}\,\mathrm{d}\boldsymbol{F}$ is given by

$$A_0 = \frac{\xi_{-d+1}}{2}\,\mathrm{d}\bar{z} + \frac{\xi_{d-1}}{2}\,\mathrm{d}z$$

Hence,

$$A_0 = i\frac{\xi_{-d+1}}{2}\,\mathrm{d}\bar{z} - i\frac{\xi_{d-1}}{2}\,\mathrm{d}z$$

Now by (5.9) we have $d * A_0 = 0$. Therefore, the equation

$$\begin{aligned} &\mathrm{d}\Omega = i * A_0, \\ &\Omega(0,0) = 0 \end{aligned} \tag{5.10}$$

has a smooth solution $\Omega : U_0 \to \boldsymbol{it}$ and, hence, $\Omega_z = \xi_{d-1}/2$, $\Omega_{\bar{z}} = -\xi_{-d+1}/2$ hold on U. If we choose the initial condition ξ° in (5.7) so that ${\xi^\circ}_d = \boldsymbol{M} \in \mathfrak{m}_1$, then $\mathrm{Ad}(\mathrm{e}^{\Omega})\boldsymbol{M}$ and ξ_d both satisfy the same system of ODE's

$$\begin{aligned} &Y_z = -\tfrac{1}{2}[Y, \xi_{d-1}] \\ &Y_{\bar{z}} = \tfrac{1}{2}[Y, \xi_{-d+1}] \\ &Y(0,0) = \boldsymbol{M} \end{aligned} \tag{5.11}$$

Therefore, $\xi_d = \mathrm{Ad}(\mathrm{e}^{\Omega})\boldsymbol{M}$ on U_0. Thus, each member of the loop $S^1 \ni \lambda \mapsto \boldsymbol{F}_\lambda$ obtained by integration of (5.8) is a Toda frame for the τ-primitive map $\psi_\lambda = \pi \circ \boldsymbol{F}_\lambda : U_0 \to \mathfrak{F}_m$. In particular, $\Omega : U_0 \to \boldsymbol{it}$ is a solution of the 2D-affine Toda field (4.17).

Remark 5.1. Note that since $[\boldsymbol{M}, \sigma(\boldsymbol{M})] \neq 0$, the null map $\Omega \equiv 0$ can not be neither a solution of (4.17) nor (5.10).

According to the Riemann mapping theorem, the connected simply connected open neighborhood U_0 of $(0,0)$ is either the complex plane $\boldsymbol{C}$ or is conformally equivalent to the open unit disc $D = \{z : |z| < 1\}$. By applying an earlier result of Sattinger it is possible to rule out the $U_0 = \boldsymbol{C}$ possibility at least for, namely, harmonic conformal maps of $U_0 \to H^n$.

Theorem 5.1. *Let M be a connected non-compact Riemann surface, and $f : M \to \boldsymbol{H}^n$ $(n > 2)$ be a minimal immersion, i.e. a harmonic conformal map. Then the universal covering space $\tilde{M}$ of M is conformaly equivalent to the open unit disc $D_1 = \{z : |z| < 1\}$.*

In particular, the theorem is true for minimal superconformal immersions of connected surfaces into H^n.

Proof. Let us suppose that $\tilde{M}$ is conformaly equivalent to $\boldsymbol{C}$. Then $\tilde{f} = f \circ \pi : \boldsymbol{C} \to \boldsymbol{H}^n$ is minimal, where $\pi : \boldsymbol{C} \to M$ is the covering map. Let $K(g)$ be the Gaussian curvature of the induced metric $g = \tilde{f}^* h$. Respect to the global isothermal coordinate z on $\boldsymbol{C}$, we have $g = \mathrm{e}^{2u}\,\mathrm{d}z\,\mathrm{d}\bar{z}$ for some smooth function u defined on the whole complex plane. In terms of u the curvature is given by $K(g) = -(1/2)4\mathrm{e}^{-2u}(2u_{z\bar{z}})$. Hence, the globally defined smooth function u must satisfy

$$\Delta u = -K(g)\mathrm{e}^{2u},$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Since $\tilde{f}$ is conformal harmonic, the Gaussian curvature satisfies $K(g) \leq -1$. Under such conditions Sattinger [13] proved that there is no solution u of the above equation defined on the whole complex plane. □

Recall the Poincaré model of H^2, $(D_1; \mathrm{d}s^2 = 4|\mathrm{d}z|^2/(1-|z|^2)^2)$. Using Theorem 5.1 we may (modulo conformal transformations of the plane) summarize the results of this section in the following.

Theorem 5.2. *Let $m \geq 2$ and $d \equiv 1 \bmod (2m)$. Recall the commuting vector fields V, $\sigma(V)$ defined on Λ_d by* (5.6).

(i) *There exists a unique solution $\xi : H^2 \to \Lambda_d$ of the ODE's system* (5.7) *with initial condition $\xi(0,0) = \xi^\circ \in \Lambda_d$, with ${\xi^\circ}_d = \boldsymbol{M} \in \mathfrak{m}_1$.*

(ii) *The Λ_1-valued one-form defined in terms of the solution ξ of* (i),

$$\alpha_\lambda = \lambda\xi_d\,\mathrm{d}z + \tfrac{1}{2}(\xi_{d-1}\,\mathrm{d}z + \xi_{-d+1}\,\mathrm{d}\bar{z}) + \lambda^{-1}\xi_{-d}\,\mathrm{d}\bar{z}$$

satisfies the Maurer–Cartan equation $\mathrm{d}\alpha_\lambda + (1/2)[\alpha_\lambda \wedge \alpha_\lambda] = 0\,\forall\lambda \in S^1$. Hence, integration of $\boldsymbol{F}_\lambda^{-1}\,\mathrm{d}\boldsymbol{F}_\lambda = \alpha_\lambda, \quad \boldsymbol{F}_\lambda(0,0) = Id\,\forall\lambda \in S^1$ gives rise to an S^1-loop of frames $\boldsymbol{F}_\lambda : H^2 \to \mathrm{SO}_o(1,2m)$ for the primitive maps $\psi_\lambda = \pi \circ \boldsymbol{F}_\lambda : H^2 \to \mathfrak{F}_m$, and consequently this produces an S^1-loop of minimal superconformal immersions $f_\lambda = P \circ \psi_\lambda : H^2 \to H^{2m}$.

(iii) *The $\boldsymbol{F}_\lambda$ are Toda frames of the primitive maps $\psi_\lambda = \pi \circ \boldsymbol{F}_\lambda : H^2 \to \mathfrak{F}_m$. Thus, there exist a solution $\Omega : H^2 \to i\boldsymbol{t}$ to the 2D-affine Toda field* Eq. (4.17).

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